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2004 J. Phys. A: Math. Gen. 37 9557

(http://iopscience.iop.org/0305-4470/37/40/016)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 9557–9571 PII: S0305-4470(04)77227-3

Relativistic supersymmetric quantum mechanics based on Klein–Gordon equation

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Received 5 March 2004
Published 22 September 2004
Online at stacks.iop.org/JPhysA/37/9557
doi:10.1088/0305-4470/37/40/016

Abstract
Witten's the non-relativistic formalism of supersymmetric quantum mechanics was based on a factorization and partnership between Schrödinger equations. We show how it accommodates a transition to the partnership between relativistic Klein–Gordon equations.

PACS numbers: 03.65.Fd, 03.65.Ca, 03.65.Ge, 11.30.Pb, 12.90.Jv

1. Introduction

The search for parallels between the non-relativistic Schrödinger equation

$$\mathrm{i}\partial_t \psi^{(\mathrm{SE})}(x,t) = \hat{h}^{(\mathrm{SE})}\psi^{(\mathrm{SE})}(x,t) \tag{1}$$

and the Klein–Gordon equation referring to relativistic kinematics in zero-spin case,

$$(\mathrm{i}\partial_t)^2\Psi^{(\mathrm{KG})}(x,t) = \hat{H}^{(\mathrm{KG})}\Psi^{(\mathrm{KG})}(x,t) \tag{2}$$

has met one of its important successes in the work by Feshbach and Villars [1] who found that a close connection between equations (1) and (2) was mediated by the degree-lowering method of Peano and Baker [2]. In such an approach, one treats the first derivative $\varphi_1^{(\mathrm{PB})}(x,t) = \mathrm{i}\partial_t\Psi^{(\mathrm{KG})}(x,t)$ of a wavefunction as an independent quantity complemented by the wavefunction itself, $\varphi_2^{(\mathrm{PB})}(x,t) = \Psi^{(\mathrm{KG})}(x,t)$. In this notation, the Klein–Gordon equation (2) may be re-written in Schrödinger-like form

$$\mathrm{i}\partial_t \begin{pmatrix} \varphi_1^{(\mathrm{PB})}(x,t) \\ \varphi_2^{(\mathrm{PB})}(x,t) \end{pmatrix} = \hat{h}^{(\mathrm{PB})} \cdot \begin{pmatrix} \varphi_1^{(\mathrm{PB})}(x,t) \\ \varphi_2^{(\mathrm{PB})}(x,t) \end{pmatrix}, \qquad \hat{h}^{(\mathrm{PB})} = \begin{pmatrix} 0 & \hat{H}^{(\mathrm{KG})} \\ 1 & 0 \end{pmatrix}. \tag{3}$$

The manifest non-Hermiticity of the relativistic Peano–Baker Hamiltonian $\hat{h}^{(\mathrm{PB})}$ in the latter equation seems to obstruct its compatibility with quantum mechanics. A new hope has

been provided by the pioneering letter by Bender and Boettcher [3] who studied some non-Hermitian non-relativistic Hamiltonians $\hat{h}^{(\mathrm{SE})}$ and emphasized that they may often possess a real, 'observable' spectrum [4]. This immediately caused a perceivable intensification of interest in all the models with real spectra which satisfy a weakened (or 'pseudo-') Hermiticity condition [5],

$$[\hat{h}^{(\mathrm{SE})}]^{\dagger} = G\hat{h}^{(\mathrm{SE})}G^{-1}, \qquad G = G^{\dagger}. \tag{4}$$

Bender's and Boettcher's reduction of the Hermitian conjugation to the mere action of the time-reversal operator $\mathcal{T}$ proved as productive as their simplifying identification of 'the metric' G with the parity operator $\mathcal{P}$. A more or less systematic description of many new $\mathcal{PT}$ symmetric models followed, and the unexpectedly robust reality of their spectra has been confirmed [6]. Still, within an emerging $\mathcal{PT}$ symmetric and/or pseudo-Hermitian quantum mechanics, thorough and lasting attention had to be paid to the unpleasant, indefinite character of the conserved pseudo-norm [7].

Progress was quick. Firstly, within a broad subset of the exactly solvable $\mathcal{PT}$ symmetric non-relativistic models the existence of a specific symmetry (called quasi-parity $\mathcal{Q}$ [8]) has been revealed, with the main capability of the elimination of the indeterminacy of the sign in the pseudo-norm. Temporarily, the existence of the quasi-parity seemed restricted to the exactly solvable models only [9]. Fortunately, in 2002, the ultimate resolution of the puzzle was described by Mostafazadeh [10] and, independently, by Bender *et al* [11]. On an abstract algebraic level, the former author imagined that an elementary clarification of the situation may be based on the fact (well known, e.g., in nuclear physics [12]) that each diagonalizable operator $\hat{h}^{(\mathrm{SE})}$ with real spectrum may be assigned *many* alternative metrics G_j. Some of them *must* be positive definite (let us denote them by G_+) in many cases of immediate interest, with a particular exemplification $G_+ = \mathcal{QP}$ using quasi-parity in the solvable examples as mentioned above. In parallel, Bender *et al* [11] introduced the very similar metric $G_+ = \mathcal{CP}$ where, in contrast to the previous case, no solvability assumption was needed. Their additional symmetry $\mathcal{C}$ carries the name of a 'charge' of the system and is defined by its spectral representation. The practical feasibility of the perturbative and/or variational constructions of $\mathcal{C}$ has been demonstrated in [13].

As long as the relativistic Klein–Gordon equation (3) is characterized by the two-by-two form of the pseudo-Hermiticity of its Hamiltonian,

$$[\hat{h}^{(\mathrm{PB})}]^{\dagger} = \mathcal{G}\hat{h}^{(\mathrm{PB})}\mathcal{G}^{-1}, \qquad \mathcal{G} = \begin{pmatrix} 0 & G \\ G & 0 \end{pmatrix} \tag{5}$$

we *know* that it also possesses an alternative positive metric $\mathcal{G}_+$ which induces its *fully consistent* probabilistic interpretation. As long as such an argument has a mere *implicit* character, Mostafazadeh offered, in his three very recent papers [14], also an *explicit* example of a sufficiently simple 'physical' metric $\mathcal{G}_+$. Choosing just the most elementary free Klein–Gordon operator $\hat{H}_0^{(\mathrm{KG})} = -\partial_x^2 + m_0^2 > 0$ he described thoroughly the properties of the resulting inner product which is conserved and *positive-definite* on the whole space of the solutions of equation (3). This means that the free Klein–Gordon equation is capable of guaranteeing the existence of the conserved norm, etc. A path towards a satisfactory and consistent relativistic quantum mechanics is open. In this sense, we feel encouraged to return to equation (3), say, with a nontrivial model interaction based on a 'minimal' replacement of the constant m_0^2 by its coordinate-dependent, 'scalar' potential or 'local effective-mass' generalization $m^2(x) = m_0^2 + V(x)$.

In what follows we intend to develop the most natural Klein–Gordon generalization of Witten's supersymmetric quantum mechanics (SUSY QM, [15, 16]). For the time being let us choose just the trivial partitions $G = I$ in equation (5) (i.e., Hermitian sub-Hamiltonians

$\hat{H}^{(\mathrm{KG})}$) and summarize our forthcoming effort as a construction which circumvents the three most serious obstacles which might mar a naive SUSY construction in the Klein–Gordon case.

(a) The spectrum may become 'too rich' for supersymmetrization purposes.
(b) The Klein–Gordon supersymmetric construction will be nontrivial because, in contrast to the traditional non-relativistic case, one of the partner equations exhibits an anomalous, 'unavoided' crossing of its energy levels.
(c) The Klein–Gordon supersymmetrization will require a new, not yet known type of factorization of the operators.

In the following, point (a) will be dealt with in section 2. We shall see there that many 'too strongly interacting' operators $\hat{h}^{(\mathrm{PB})}$ may possess complex and, hence, physically meaningless eigenvalues. This obstacle is, fortunately, trivial since in such a regime the forces in action must be very strong so that one encounters a phenomenologically motivated limit of validity of quantum mechanics. One is forced to use many more degrees of freedom which may be made available, most typically, in quantum field theory.

The study of point (b) will be initiated in section 3 where we recollect the main steps of supersymmetrization in a non-relativistic context and remind the reader that the supersymmetry requires that the ground-state energy vanishes. In the relativistic Klein–Gordon setting where the first unavoided crossing of quantum levels occurs precisely at the vanishing energy, this requirement becomes highly nontrivial.

The core of our present message concerns point (c) and will be discussed in section 4. We shall propose a new type of a factorization of our Hamiltonians, the feasibility of which will be achieved via an additional partitioning of our Hilbert space(s). We shall emphasize the intimate relationship of the properties of our new factorization to the above-mentioned existence of the level crossings, connected also to the numerical concept of the Jordan blocks in non-Hermitian matrices [17]. The summary of our construction will be given in section 5.

2. Non-Hermiticity and spectra of Klein–Gordon equations

2.1. Auxiliary partitioning of Hilbert space

Let us assume that the operator $\hat{h}^{(\mathrm{PB})} \neq (\hat{h}^{(\mathrm{PB})})^{\dagger}$ in the Peano–Baker form (3) of our Klein–Gordon equation (2) does not depend on time and remains 'regular', $\hat{h}^{(\mathrm{PB})} = \hat{h}^{(\mathrm{PB})}_{(\mathrm{R})}$ having no pathological complex or vanishing eigenvalues. Then, the standard Fourier-type ansatz for wavefunctions,

$$\varphi_1^{(\mathrm{PB})}(x,t) = \int U(x,E)\,\mathrm{e}^{-\mathrm{i}Et}\,\mathrm{d}\mu(E), \qquad \varphi_2^{(\mathrm{PB})}(x,t) = \int D(x,E)\,\mathrm{e}^{-\mathrm{i}Et}\,\mathrm{d}\mu(E) \tag{6}$$

reduces equation (2) to its linear algebraic eigenvalue version

$$\begin{pmatrix} 0 & \hat{H}^{(\mathrm{KG})} \\ I & 0 \end{pmatrix} \cdot \begin{pmatrix} U(x,E) \\ D(x,E) \end{pmatrix} = E \begin{pmatrix} U(x,E) \\ D(x,E) \end{pmatrix}. \tag{7}$$

The solutions have to be sought in the relativistic Feshbach–Villars Hilbert space composed of two equivalent subspaces, $\mathcal{R}_{(\mathrm{R})} = \mathcal{H}_{(\mathrm{R})} \oplus \mathcal{H}_{(\mathrm{R})}$. Their components $U(x,E)$ ('up') and $D(x,E)$ ('down') belong to the identical subspaces $\mathcal{H}_{(\mathrm{R})}$ with elements which will be denoted by the curly kets $|\cdot\}$. In this notation, the second row of the partitioned equation (7) implies that $|U\} = E|D\}$. Formally, this solves equation (7) by its reduction to the equation

$$\hat{H}^{(\mathrm{KG})}\big|D_n^{(\pm)}\big\} = \varepsilon_n\big|D_n^{(\pm)}\big\}, \qquad n = 0, 1, \ldots \tag{8}$$

in $\mathcal{H}_{(\mathrm{R})}$. Under our above assumption that $\varepsilon_n > 0$, the original energies form pairs $E_n^{(\pm)} = \pm\sqrt{\varepsilon_n}$ and remain real. In many cases, the set of kets $\big|D_n^{(+)}\big\}\big\}$ (equal, up to a

multiplicative factor, to $|D_n^{(-)}\}$) will be complete in $\mathcal{H}_{(\mathrm{R})}$. In contrast, the 'up' components $\big|U_n^{(\pm)}\big\} = E_n^{(\pm)}\big|D_n^{(\pm)}\big\}$ depend on the superscript and define the *pairs* of independent eigenvectors of equation (7) numbered by the sign superscript *and* integers n,

$$\hat{h}^{(\mathrm{PB})}|n^{(\pm)}\rangle = E_n^{(\pm)}|n^{(\pm)}\rangle, \qquad |n^{(\pm)}\rangle = \begin{pmatrix} \pm\sqrt{\varepsilon_n}\big|D_n^{(\pm)}\big\} \\ \big|D_n^{(\pm)}\big\} \end{pmatrix}. \tag{9}$$

We shall often assume that these kets span the whole relativistic Hilbert space $\mathcal{R}$ (see the appendix giving more details).

2.2. An exactly solvable illustrative example

For a class of the model Klein–Gordon operators

$$H^{(\mathrm{KG})} = -\partial_x^2 + m^2(x), \qquad x \in \mathbb{R} \tag{10}$$

where $m_0 = m(\pm\infty)$ is the free asymptotic mass and $c = \hbar = 1$, the key assumption $\varepsilon_n > 0$ of our preceding paragraph is easily satisfied for sufficiently smooth and small variations of the 'effective mass' $m^2(x)$ at the finite coordinates $x \in \mathbb{R}$. Such a choice of the model enables us to discuss some of the consequences of making this coordinate dependence less and less smooth and/or bounded. For this purpose, let us pick up one of the most elementary models with attraction,

$$m^2(x) = m_0^2 + \frac{B^2 - A^2 - A\omega}{\cosh^2 \omega x} + \frac{B(2A+\omega)\sinh \omega x}{\cosh^2 \omega x} \tag{11}$$

which is exactly solvable and shape invariant [16]. Its exact solvability means that its bound-state solutions generated by our Klein–Gordon equation (7) exist at (a finite set of) the energies

$$E_n^{(\pm)} = \pm\sqrt{m_0^2 - (A - n\omega)^2}, \qquad n = 0, 1, \ldots, n_{\max}, \qquad n_{\max} = \mathrm{entier}[A/\omega]. \tag{12}$$

We may distinguish among the three regimes. In the first one with $0 < A < m_0$, the strength of the force is weak and we have well-behaved eigenvalues at all the admissible indices n. In the second, strong-coupling regime with $0 < m_0 < A$, a few low-lying states suffer a collapse and acquire, formally, complex energies, i.e., $\mathrm{Im}\big(E_0^{(\pm)}\big) \neq 0$, $\mathrm{Im}\big(E_1^{(\pm)}\big) \neq 0$ etc. We have certainly left the domain of quantum mechanics.

At the boundary point, we have $A = m_0 > 0$. During a limiting transition $\varepsilon_0 \to 0^+$ the two smallest eigenvalues $E_0^{(\pm)}$ merge and vanish. Under the name of an exceptional point (EP, [18]) the energy $E_0 = 0$ still represents a valid bound state because after one abbreviates

$$y = y(x) = \sinh \omega x, \qquad s = A/\omega > 0, \qquad t = B/\omega, \tag{13}$$

the wavefunctions pertaining to the set of the energies $E_n^{(\pm)}$ may *all* be written in closed form in terms of Jacobi polynomials,

$$\langle x|D_n\} = \mathrm{i}^n[1 + y(x)^2]^{-s/2}\exp[-t\arctan y(x)]P_n^{(\mathrm{i}t-s-1/2,-\mathrm{i}t-s-1/2)}[\mathrm{i}y(x)]. \tag{14}$$

In spite of appearances, all of them remain normalizable at all $s > 0$ so that also the solution with $E_0^{(\pm)} = \varepsilon_0 = 0$ (in fact, just a single state) does not represent any exception. All the wavefunctions appear to be purely real and after we insert the abbreviation $y(x) = \sinh \omega x$ we get

$$\langle x|D_0\} = (1 + \sinh^2 \omega x)^{-s/2}\exp(-t\,\mathrm{arctan\,sinh}\,\omega x), \tag{15}$$

$$\langle x|D_1\} = (1 + \sinh^2 \omega x)^{-s/2}\exp(-t\,\mathrm{arctan\,sinh}\,\omega x)\left(\frac{2s-1}{2}\sinh \omega x - t\right), \qquad \ldots. \tag{16}$$

One should be fully aware that in contrast to the robust character of the wavefunctions, the vanishing energy of the EP state at $n = 0$ becomes highly sensitive to a perturbation of the potential. An arbitrarily small downward shift of $m^2(x)$ converts the degenerate ground-state energy $E_0^{(\pm)} = 0$ into a complex conjugate pair of purely imaginary values, leading to an obvious physical instability.

2.3. Action of the non-Hermitian $\hat{h}^{(\mathrm{PB})}$ to the left

Because of the non-Hermiticity (5) of $\hat{h}^{(\mathrm{PB})}$, it is convenient to complement equation (7) by its analogue where the Hermitian conjugation of the operator $\hat{h}^{(\mathrm{PB})}$ is considered,

$$\begin{pmatrix} 0 & I \\ [\hat{H}^{(\mathrm{KG})}]^\dagger & 0 \end{pmatrix} \cdot \begin{pmatrix} L^*(x,E) \\ R^*(x,E) \end{pmatrix} = E^* \begin{pmatrix} L^*(x,E) \\ R^*(x,E) \end{pmatrix}. \tag{17}$$

In what follows, we shall prefer the more natural form of the same equation where the operator $\hat{h}^{(\mathrm{PB})}$ is simply assumed to act to the left, i.e., where the usual transposed vector-times-matrix multiplication convention is used,

$$[L(x,E),\, R(x,E)] \cdot \begin{pmatrix} 0 & \hat{H}^{(\mathrm{KG})} \\ 1 & 0 \end{pmatrix} = E \cdot [L(x,E),\, R(x,E)]. \tag{18}$$

We shall employ the double-bra-like symbols $\{\{\cdot|$ for both the 'left' $L(x,E)$ and the 'right' $R(x,E)$ functions in the dual of $\mathcal{H}$. *Mutatis mutandis* we arrive at the auxiliary definition of $\{\{R_n^{(\pm)}| = E_n^{(\pm)}\{\{L_n^{(\pm)}|$ and reduce equations (17) and (18) to the standard eigenvalue problem $[\hat{H}^{(\mathrm{KG})}]^\dagger |L_n^{(\pm)}\}\} = (\varepsilon_n)^* |L_n^{(\pm)}\}\}$, i.e., under our present conventions,

$$\{\{L_n^{(\pm)}|\hat{H}^{(\mathrm{KG})} = \varepsilon_n \{\{L_n^{(\pm)}|, \qquad n = 0, 1, \ldots. \tag{19}$$

The new solutions $\{\{L_n^{(\pm)}|$ emerge at the same eigenvalues ε_n as above. This means that we may denote all the doubly partitioned left row eigenvectors in the (dual of the) larger Hilbert space $\mathcal{R}$ by the double-bra symbol,

$$\langle\langle n^{(\pm)}| = \big(\{\{L_n^{(\pm)}|, \pm\sqrt{\varepsilon_n}\{\{L_n^{(\pm)}|\big). \tag{20}$$

Further merits of this compactified notation are summarized in the appendix.

3. Non-relativistic supersymmetric quantum mechanics

Non-relativistic supersymmetric quantum mechanics (SUSY QM, cf its comprehensive review [16]) may be defined as a formalism where the operator of energy (i.e., an essentially self-adjoint Hamiltonian $H = H^\dagger$ defined in a space $\mathcal{S}$) happens to coincide with a generator of a graded Lie algebra. For example, we may incorporate such a H in the $sl(1|1)$ multiplication table

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \qquad \{Q, Q^\dagger\} = H, \qquad [H, Q] = [H, Q^\dagger] = 0 \tag{21}$$

containing commutators as well as anticommutators. The other two generators (so-called 'supercharges' Q and $Q^\dagger$) remain non-Hermitian, i.e., unobservable. The potential appeal of such a non-Lie model in physics has been revealed, i.e., by Witten [15] who claimed that the analyses of similar systems (tractable also as field theories in zero dimensions in particle physics) could clarify the puzzling experimental absence of 'supermultiplets' containing both bosons *and* fermions. For this reason, it sounds like a paradox that while the methodical impact of the oversimplified SUSY QM on experimental physics remained

virtually negligible, the supersymmetry using superalgebra $sl(1|1)$ offers a lasting inspiration of theoretical developments in quantum mechanics [19].

The representations of the superalgebra $sl(1|1)$ were extended to non-Hermitian Hamiltonians [20]. A particularly promising continuation of this direction of research might eventually be related to the relativistic extension of SUSY QM based on the use of Klein–Gordon equations [21]. Indeed, the more traditional Hermitian constructions seem both too formal and too narrow. Too formal because almost the whole story of the traditional SUSY QM may be re-told as an application of Schrödinger's factorization method [22], and too narrow because even the simplest relativistic spin-one-half Dirac-equation examples (cf their sketchy review in section 11 of [16]) do not seem to fit into the general scheme. In what follows we shall try to address and weaken both of these objections.

3.1. Representations of superalgebra sl(1|1)

The validity of the mixed commutation and anticommutation relations (21) in SUSY QM is guaranteed by the two-by-two matrix choice of all the generators in question,

$$Q = \begin{pmatrix} 0 & 0 \\ \hat{b} & 0 \end{pmatrix}, \qquad Q^\dagger = \begin{pmatrix} 0 & \hat{b}^\dagger \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} \hat{h}_{\rm L} & 0 \\ 0 & \hat{h}_{\rm R} \end{pmatrix}. \tag{22}$$

The upper (= left) sub-Hamiltonian $\hat{h}_{\rm L}$ and its 'right' partner $\hat{h}_{\rm R}$ are both acting in separate Hilbert spaces $\mathcal{H}_{({\rm L,R})}$ such that $\mathcal{S} = \mathcal{H}_{({\rm L})} \oplus \mathcal{H}_{({\rm R})}$. In these subspaces, both the sub-Hamiltonians are essentially self-adjoint and may be expected to possess a spectral representation, i.e., formally, $\hat{h}_{\rm L,R} = \sum_n |n_{({\rm L,R})}\rangle \varepsilon^{(n)}_{({\rm L,R})} \langle n_{({\rm L,R})}|$. In addition, both of them must be factorizable, $\hat{h}_{\rm L} = \hat{b}^\dagger \hat{b}$ and $\hat{h}_{\rm R} = \hat{b}\hat{b}^\dagger$. This is one of the most important consequences of the SUSY postulate (21). For the same reason, the two sets of wavefunctions are interconnected by the chain rules

$$|n_{({\rm R})}\rangle = {\rm const} \cdot \hat{b}|(n+1)_{({\rm L})}\rangle, \qquad n = 0, 1, \ldots \tag{23}$$

$$|(n+1)_{({\rm L})}\rangle = {\rm const}' \cdot \hat{b}^\dagger |n_{({\rm R})}\rangle, \qquad n = 0, 1, \ldots \tag{24}$$

and in the regime of the so-called unbroken supersymmetry we may often deduce that the sub-spectra are, up to the ground state, the same,

$$0 = \varepsilon^{(0)}_{({\rm L})} < \varepsilon^{(0)}_{({\rm R})} = \varepsilon^{(1)}_{({\rm L})} < \varepsilon^{(1)}_{({\rm R})} = \varepsilon^{(2)}_{({\rm L})} < \cdots . \tag{25}$$

Although the L- and R-bases may be different in principle, we may put

$$\hat{b} = \sum_m |m_{({\rm R})}\rangle \cdot \beta_m \cdot \langle (m+1)_{({\rm L})}|, \qquad \hat{b}^\dagger = \sum_p |(p+1)_{({\rm L})}\rangle \cdot \beta^*_p \cdot \langle p_{({\rm R})}| \tag{26}$$

and define easily also the basis and spectral representation of our self-adjoint super-Hamiltonian H in a larger Hilbert space $\mathcal{S}$.

In many applications, the basis is not employed and the factorization recipe is specified in terms of the following linear differential operators:

$$\hat{b} = \partial_x + W(x), \qquad \hat{b}^\dagger = -\partial_x + W(x), \qquad \hat{h}_{\rm L/R} = -\partial_x^2 + W^2(x) \mp W'(x). \tag{27}$$

Here, the so-called superpotential $W(x)$ may be derived from an arbitrary auxiliary input ground-state wavefunction,

$$W(x) = -\frac{\psi_0'(x)}{\psi_0(x)}, \qquad \psi_0(x) \in \mathcal{H}_{({\rm L})}. \tag{28}$$

For illustration one may select the harmonic-oscillator ground-state $\psi_0^{({\rm HO})}(x) \sim \exp(-x^2/2)$ giving the linear $W^{({\rm HO})}(x) = x$ and the two partner Hamiltonians $\hat{h}^{({\rm HO})}_{\rm L/R} = -\partial_x^2 + x^2 \mp 1$. Their pair forms the desired super-Hamiltonian $H^{({\rm HO})}$ with both the bosonic *and* fermionic features.

3.2. Example: bosonic plus fermionic harmonic oscillator

The simultaneous use of the language of bases and differential operators is useful. In particular, our explicit knowledge of the HO basis may help us to clarify the essence of the factorization property since in the infinite-dimensional matrix notation with

$$\left|0^{(\mathrm{HO})}_{(\mathrm{L})}\right\rangle = \left|0^{(\mathrm{HO})}_{(\mathrm{R})}\right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \qquad \left|1^{(\mathrm{HO})}_{(\mathrm{L})}\right\rangle = \left|1^{(\mathrm{HO})}_{(\mathrm{R})}\right\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \qquad \dots \tag{29}$$

we may simply put

$$\hat{b} = \begin{pmatrix} 0 & \beta_0 & 0 & \dots & \\ 0 & 0 & \beta_1 & 0 & \dots \\ 0 & 0 & 0 & \ddots & \end{pmatrix}, \qquad \hat{b}^\dagger = \begin{pmatrix} 0 & \dots & & \\ \beta_0^* & 0 & \dots & \\ 0 & \beta_1^* & 0 & \dots \\ 0 & 0 & \ddots & \end{pmatrix} \tag{30}$$

and choose

$$\beta_n = \sqrt{\varepsilon^{(n+1)}_{(\mathrm{L})}} = \sqrt{\varepsilon^{(n)}_{(\mathrm{R})}}. \tag{31}$$

This facilitates the verification of the SUSY-based isospectrality (25),

$$\left\{\varepsilon^{(n)}_{(\mathrm{L})}\right\} = \left\{E^{(\mathrm{HO})}_{\mathrm{L}}\right\} = \{0, 2, 4, 6, \dots\}, \qquad \left\{\varepsilon^{(n)}_{(\mathrm{R})}\right\} = \left\{E^{(\mathrm{HO})}_{\mathrm{R}}\right\} = \{2, 4, 6, \dots\}. \tag{32}$$

Also the physics 'hidden' besides the SUSY mathematics finds an impressive Fock-space re-interpretation (as described in section 2.1 of review [16]) in the full Hilbert space $\mathcal{S}^{(\mathrm{HO})}$ spanned by the two-index ket vectors

$$|0,0\rangle = \begin{pmatrix} |0\rangle \\ 0 \end{pmatrix}, \qquad |0,1\rangle = \begin{pmatrix} 0 \\ |0\rangle \end{pmatrix}, \qquad |1,0\rangle = \begin{pmatrix} |1\rangle \\ 0 \end{pmatrix}, \qquad |1,1\rangle = \begin{pmatrix} 0 \\ |1\rangle \end{pmatrix} \tag{33}$$

etc, i.e., vectors $|n_b, m_f\rangle$ where the non-negative integer $n_b = 0, 1, 2, \dots$ may be understood as a number of bosons in the system while the absence/presence of a fermion (which obeys the Pauli exclusion principle) is characterized by the second index with the mere two admissible values $n_f = 0$ and $n_f = 1$. In the other words, the creation or annihilation of a boson is mediated by $\hat{b}^\dagger$ or $\hat{b}$, respectively, while the operators of the creation and annihilation of the fermion are just the elementary two-by-two matrices

$$F^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{34}$$

which enter also the above definitions of the supercharges, $Q = F^\dagger \cdot \hat{b}$ while $Q^\dagger = \hat{b}^\dagger \cdot F$. Marginally, let us note that in the spirit of the simplified notation of equation (29) we may also write

$$|0,0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ \dots \\ 0 \\ \vdots \end{pmatrix}, \qquad |0,1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \dots \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \qquad |1,0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ \dots \\ 0 \\ \vdots \end{pmatrix}, \qquad |1,1\rangle = \begin{pmatrix} 0 \\ \vdots \\ \dots \\ 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} \tag{35}$$

etc.

4. Relativistic representations of $sl(1|1)$

4.1. Pseudo-Hermitian $\hat{h}^{(\mathrm{PB})}_{(\mathrm{L,R})}$ and their refined partitioning

In the spirit of our introductory considerations and in a way prepared by the preceding text, let us now replace the $sl(1|1)$ commutation/anticommutation rules (21) by their less symmetric version

$$\{Q, Q\} = \{\tilde{Q}, \tilde{Q}\} = 0, \qquad [H, Q] = [H, \tilde{Q}] = 0, \qquad \{Q, \tilde{Q}\} = H \neq H^{\dagger}. \tag{36}$$

The validity of this multiplication table may be guaranteed via a slight modification of equation (22),

$$Q = \begin{pmatrix} 0 & 0 \\ \hat{a} & 0 \end{pmatrix}, \qquad \tilde{Q} = \begin{pmatrix} 0 & \hat{c} \\ 0 & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} \hat{h}_{\mathrm{L}} & 0 \\ 0 & \hat{h}_{\mathrm{R}} \end{pmatrix}. \tag{37}$$

The weakening of the Hermiticity assumption means that we must re-analyse the SUSY-induced factorization conditions in the Hilbert space $\mathcal{S} = \mathcal{R}_{(\mathrm{L})} \oplus \mathcal{R}_{(\mathrm{R})}$,

$$H = \begin{pmatrix} \hat{h}_{\mathrm{L}} & 0 \\ 0 & \hat{h}_{\mathrm{R}} \end{pmatrix} = \begin{pmatrix} \hat{c}\hat{a} & 0 \\ 0 & \hat{a}\hat{c} \end{pmatrix}. \tag{38}$$

We shall try to satisfy it via a refined partitioning of both the supercharges and of the related non-Hermitian sub-Hamiltonians which act in the two equivalent subspaces $\mathcal{R} = \mathcal{R}_{(\mathrm{L,R})}$ of $\mathcal{S}$,

$$\hat{a} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \qquad \hat{c} = \begin{pmatrix} 0 & \gamma \\ \delta & 0 \end{pmatrix} \tag{39}$$

$$\hat{h}_{\mathrm{R}} = \hat{c}\hat{a} = \begin{pmatrix} 0 & \gamma\beta \\ \delta\alpha & 0 \end{pmatrix}, \qquad \hat{h}_{\mathrm{L}} = \hat{a}\hat{c} = \begin{pmatrix} 0 & \alpha\gamma \\ \beta\delta & 0 \end{pmatrix}. \tag{40}$$

In addition to this algebra, one should return to the differential operators, imagining that a new physical meaning must also be assigned to all the new symbols. In the rest of this paper we shall insert the relativistic, Klein–Gordon differential form of the sub-Hamiltonians and show that and how all this scheme works with the right and regular $\hat{h}_{(\mathrm{R})} = \hat{h}^{(\mathrm{PB})}_{(\mathrm{R})}$ and with its left (and, as we shall see, irregular and appropriately regularized) partner $\hat{h}_{(\mathrm{L})} = \left[\hat{h}^{(\mathrm{PB})}_{(\mathrm{L})}\right]_{(\mathrm{reg.})}$.

4.2. Factorizations of the regular $\hat{h}^{(\mathrm{PB})}_{(\mathrm{R})}$

4.2.1. Algebraic factorizations in eigenbases. Before moving to the differential-operator constructions, let us once more switch to the linear-algebraic and matrix notation, first employed in subsection 3.2 and then described in full detail in the appendix. As long as we demand, first of all, that $\hat{h}^{(\mathrm{PB})}_{(\mathrm{R})} = \hat{a}\hat{c}$ we have to deal, due to equation (40), with the following two requirements,

$$\hat{H}^{(\mathrm{KG})}_{(\mathrm{R})} = \alpha \cdot \gamma > 0, \qquad \beta \cdot \delta = I. \tag{41}$$

Recollecting our experience with the non-relativistic cases we may now try to reproduce $\hat{H}^{(\mathrm{KG})}_{(\mathrm{R})}$, on the basis of its own eigenvectors, as a diagonal matrix with elements $\varepsilon_{(\mathrm{R})n}$. Most easily, this will follow from the most elementary choice of the elements

$$a_n = c_n = \sqrt{\varepsilon_{(\mathrm{R})n}}, \qquad n = 0, 1, \ldots \tag{42}$$

in

$$\alpha = \begin{pmatrix} 0 & a_0 & 0 & \ldots & \\ 0 & 0 & a_1 & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \gamma = \begin{pmatrix} 0 & \ldots & & \\ c_0 & 0 & \ldots & \\ 0 & c_1 & 0 & \ldots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{43}$$

The second identity in equation (41) will then be satisfied when we postulate, say,

$$\delta = \begin{pmatrix} 0 & \dots & & \\ d_0 & 0 & \dots & \\ 0 & d_1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1/d_0 & 0 & \dots & \\ 0 & 0 & 1/d_1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{44}$$

These rules must remain compatible with the rest of equation (40) of course.

4.2.2. The differential-operator factorizations. Besides the harmonic-oscillator illustration of the non-relativistic SUSY QM (cf subsection 3.2) we could have used the so-called Scarf example which is also described thoroughly in the review [16] and which may be derived from a slightly more complicated superpotential

$$W(x) = \frac{A\sinh\omega x + B}{\cosh\omega x}. \tag{45}$$

We preferred to recall a slightly modified version of this alternative solvable example sooner, namely, in the role of an explicit sample of the Klein–Gordon operator in subsection 2.2. The reason was that the Scarf potential may be treated as a bounded and arbitrarily small perturbation of the current free Klein–Gordon field with no interaction at all. Now we might add a comment that after a transfer of the Scarf model to the non-relativistic SUSY QM, the 'right' partner potential in the general recipe (27) is obtained simply by the replacement of the 'left' parameter A (in the explicit formula (11) as well as in the subsequent solutions) by $A - \omega$. This property is called shape invariance, and its thorough discussion may be found in [16].

On this background one can quickly deduce the most important part of the explicit *differential* version of the *algebraic* factorization rules (41). It is given by the innovated, *relativistic* formulae

$$\alpha = \beta = \partial_x + W(x), \qquad \gamma = -\partial_x + W(x), \qquad \hat{H}^{(\mathrm{KG})}_{\mathrm{L/R}} = -\partial_x^2 + W^2(x) \mp W'(x). \tag{46}$$

The concept of shape invariance is transferred to the relativistic context without any perceivable change. The only *real* difference concerns the Green function operator δ which is defined by the second identity in (41). It may have a more complicated integral-operator form but in the factorization of the operator $\hat{h}^{(\mathrm{PB})}_{(\mathrm{R})}$ its role (of a right pseudo-inverse of β) is purely formal.

4.3. The final regularization and factorization of $\hat{h}^{(\mathrm{PB})}_{(\mathrm{L})}$

The possible presence of an exceptional point $\varepsilon_{(\mathrm{L})0} = 0$ in the spectrum of the operators $\hat{h}^{(\mathrm{PB})}_{(\mathrm{L})}$ was detected in subsection 2.2. As a consequence we have to be prepared to modify the whole picture and to employ, if necessary, a transition to the regularized Hamiltonians $\left[\hat{h}^{(\mathrm{PB})}_{(\mathrm{L})}\right]_{(\mathrm{reg.})}$ which would possess a more consistent physical interpretation. Keeping this in mind, we may recollect subsection 4.1 and expect that the 'left' operator $\hat{h}^{(\mathrm{PB})}_{(\mathrm{L})}$ or rather $\left[\hat{h}^{(\mathrm{PB})}_{(\mathrm{L})}\right]_{\mathrm{reg.}} = \hat{c}\hat{a}$ acting in $\mathcal{R}_{(\mathrm{L})}$ will be *the* desired SUSY partner of the above 'right' and regular operator $\hat{h}^{(\mathrm{PB})}_{(\mathrm{R})} = \hat{a}\hat{c}$ defined in $\mathcal{R}_{(\mathrm{R})}$. The respective spectra $\{\varepsilon_{(\mathrm{L,R})n}\}$ of these two operators must be related by the isospectrality relation (25),

$$\varepsilon_{(\mathrm{L})(n+1)} = \varepsilon_{(\mathrm{R})n}, \qquad n = 0, 1, \dots, \qquad \varepsilon_{(\mathrm{L})0} = 0. \tag{47}$$

The former partner operator is defined by the product formula

$$\left[\hat{h}^{(\mathrm{PB})}_{(\mathrm{L})}\right]_{(\mathrm{reg.})} = \hat{c}\hat{a} = \begin{pmatrix} 0 & \gamma\beta \\ \delta\alpha & 0 \end{pmatrix} \tag{48}$$

in the Hilbert subspace $\mathcal{R}_{(\mathrm{L})} = \mathcal{H}_{(\mathrm{U})} \oplus \mathcal{H}_{(\mathrm{D})}$ of the whole space $\mathcal{S} = \mathcal{R}_{(\mathrm{L})} \oplus \mathcal{R}_{(\mathrm{R})}$ where the supersymmetry is to be represented. This means that we have to complement the above relations (41) by the last two missing factorization rules in the 'smallest' Hilbert spaces $\mathcal{H}$. The first one gives the infinite-dimensional matrix formula for the product $\gamma \cdot \beta = \hat{H}^{(\mathrm{KG})}_{(\mathrm{L})}$ in its own eigenbasis,

$$\begin{pmatrix} 0 & \dots & & \\ c_0 & 0 & \dots & \\ 0 & c_1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 0 & 1/d_0 & 0 & \dots \\ 0 & 0 & 1/d_1 & \dots \\ 0 & 0 & 0 & \ddots \\ \vdots & \ddots & & \ddots \end{pmatrix} = \begin{pmatrix} \varepsilon_{(\mathrm{L})0} & 0 & \dots & \\ 0 & \varepsilon_{(\mathrm{L})1} & 0 & \dots \\ 0 & 0 & \varepsilon_{(\mathrm{L})2} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{49}$$

We see that we have to fix $\varepsilon_{(\mathrm{L})0} = 0$ while the consistency of our considerations follows from the resulting $\varepsilon_{(\mathrm{L})n} = c_{n-1}/d_{n-1} = \varepsilon_{(\mathrm{R})(n-1)}$ at $n = 1, 2, \dots$. Once we choose $d_n \cdot a_n = 1$, the second matrix product evaluates to the projector

$$\delta \cdot \alpha = \begin{pmatrix} 0 & \dots & & \\ d_0 & 0 & \dots & \\ 0 & d_1 & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \cdot \begin{pmatrix} 0 & a_0 & 0 & \dots & \\ 0 & 0 & a_1 & 0 & \dots \\ 0 & 0 & 0 & a_2 & \ddots \\ \vdots & \ddots & & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \tag{50}$$

expressible, in the notation of the appendix, as a perturbed unit operator,

$$\Pi = \begin{pmatrix} 0 & 0 & \dots & \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} = I - |D_0\} \frac{1}{\{\{L_0|D_0\}} \{\{L_0|. \tag{51}$$

We might conclude that a full *mathematical* analogy with the non-relativistic SUSY QM of [16] is being achieved at the cost of the regularization $I \to \Pi$ of our Hamiltonian's action in $\mathcal{R}_{(\mathrm{L})}$. On the same purely mathematical level, the limiting transition $\varepsilon_{(\mathrm{L})0} \to 0$ has a natural consequence $|U_0\} \to 0$ and, hence, $\varphi_1(x, t) \to 0$. This might have been expected in advance because the related wavefunction $\varphi_2(x, t)$ cannot depend on time at $E = \sqrt{\varepsilon_0} = 0$. For this reason, the original solutions $\varphi^{(\mathrm{PB})}_{1,2}(x)$ remain unchanged when the unit operator I in the relation (7) is replaced by the projector Π.

Vice versa, *before* the regularization $I \to \Pi$ in the operator $\hat{h}^{(\mathrm{PB})}_{(\mathrm{L})}$, both the sets of its respective right and left eigenvectors $|n^{(\pm)}\rangle$ and $\langle\langle n^{(\pm)}|$ were not complete because the two $n = 0$ kets $|0^{(\pm)}\rangle$ (as well as bras $\langle\langle 0^{(\pm)}|$) ceased to be independent in the limit $\varepsilon_0 \to 0$. *After* the regularization $I \to \Pi$ there emerge the new $n = 0$ independent pairs of eigenvectors

$$|0^{(\pm)}\rangle = \begin{pmatrix} \pm\sqrt{\eta}|D_0\} \\ |D_0\} \end{pmatrix}, \qquad \langle\langle 0^{(\pm)}| = \left(\{\{L_0|, \pm\sqrt{\eta}\{\{L_0|\right) \tag{52}$$

(with any $\eta > 0$) which make the sets of eigenvectors complete in $\mathcal{R}$. Then we may freely apply all the formulae provided by the appendix in the regular case. The new 'missing'

vectors (52) do not play any important role after all. They do not even enter the spectral representation of the block $\hat{H}^{(\mathrm{PB})}_{(\mathrm{L})}$ or projector Π in our factorized Klein–Gordon SUSY partner $\left[\hat{h}^{(\mathrm{PB})}_{(\mathrm{L})}\right]_{(\mathrm{reg.})}$ of $\hat{h}^{(\mathrm{PB})}_{(\mathrm{R})}$ at all.

5. Summary

Our present extension of the formalism of SUSY QM to the relativistic domain is methodically promising and conceptually transparent. It preserves a close similarity between the Klein–Gordon equation (3) and the usual Schrödinger equation (1). We found a technical key to their parallelism in a refined partitioning of the Hilbert space in the relativistic case. We also avoided the main obstacle of the relativistic SUSY QM construction which lies in the difficult *physical* interpretation of the vanishing ground-state or exceptional-point eigenvalue $\varepsilon_{(\mathrm{L})0} = 0$.

In the context of the recent studies of the Klein–Gordon equation, the emergence of the EP difficulty is new. The reason is that a strong attraction must be present in our $\hat{H}^{(\mathrm{KG})}_{(\mathrm{L})}$ in order to produce the eigenvalue which lies so deeply below the boundary of the continuous spectrum $m^2(\infty) = m_0^2$. The exceptional vanishing value of $\varepsilon_{(\mathrm{L})0}$ corresponds to the strength of interaction where a small additional perturbation is already able to produce a collapse or, in the language of the relativistic quantum field theory, a spontaneous pair-creation, etc.

One of the key merits of non-relativistic SUSY QM is the explanation of the origin of the exact solvability of many traditional Schrödinger equations. In the future, our SUSY construction might play a similar mathematical role in the relativistic context. Its key features parallel the list of obstacles in our introductory section:

- The range of our Klein–Gordon SUSY models is constrained to the 'no-collapse' regime where the energies remain real. We cannot describe effects like the spontaneous creation of particle–antiparticle pairs of course. The natural boundary of our formalism is clearly marked by the spontaneous complexification of the energy pairs.
- Phenomenological aspects of our present SUSY formalism are transparent and similar to their non-relativistic predecessors. A notable exception concerns the unclear interpretation of the vanishing energies, circumvented here by a regularization of the 'left' sub-Hamiltonians $\hat{h}^{(\mathrm{PB})}_{\mathrm{L}}$.
- The mathematical feasibility of the factorization of our present relativistic Klein–Gordon Hamiltonians has been shown rendered possible by the *same* formal regularization as above. This coincidence may be understood as a quantum mechanical analogue of consistency of the high-energy cut-off in quantum field theory.

Marginally, let us point out the possible methodical relevance of the present regularization of the level crossing at $\varepsilon_0 = 0$ for an improvement of our future understanding of some models in non-quantum domains of physics where the very similar 'exceptional-point' singularities have been observed inside the intervals of variability of relevant parameters [18, 23].

Acknowledgment

Work partially supported by the grant no A 1048302 of GA AS CR.

Appendix. Biorthogonality and completeness

In the notation of section 2 (omitting, temporarily, the 'pathologically' vanishing $\varepsilon_0 = 0$), the results of the solution of the two equations

$$\hat{h}^{(\mathrm{PB})}|n^{(\pm)}\rangle = E_n^{(\pm)}|n^{(\pm)}\rangle, \qquad \langle\langle n^{(\pm)}|\hat{h}^{(\mathrm{PB})} = E_n^{(\pm)}\langle\langle n^{(\pm)}| \tag{A.1}$$

may be well assumed to span the Hilbert space $\mathcal{R}$. As long as the superscripts $^{(\pm)}$ may enter just the norms, we may abbreviate $\left|D_n^{(\pm)}\right\} = \nu_n^{(\pm)}|D_n\}$ and $\left|L_n^{(\pm)}\right\}\} = \kappa_n^{(\pm)}|L_n\}\}$ and treat the two reduced equations

$$\hat{H}^{(\mathrm{KG})}|D_n\} = \varepsilon_n|D_n\}, \qquad \{\{L_n|\hat{H}^{(\mathrm{KG})} = \varepsilon_n\{\{L_n|, \qquad n = 0, 1, \ldots \tag{A.2}$$

as the source of the two complete sets in $\mathcal{H}$.

A.1. Smaller Hilbert spaces $\mathcal{H}$

Functions defined by the two equations (A.2) may be used to pre-multiply the partner equation. The subsequent subtraction of the resulting matrix elements may be easily checked to give the rule

$$0 = (\varepsilon_m - \varepsilon_n)\{\{L_m|D_n\}. \tag{A.3}$$

This is to be read as a bi-orthogonality relation in the regular space $\mathcal{H}_{(\mathrm{R})}$ where $\varepsilon_0 > 0$. The overlaps $\{\{L_m|D_n\}$ must necessarily vanish whenever the energies remain non-degenerate or at any pair of subscripts such that $\varepsilon_m \neq \varepsilon_n$. For the sake of a simplification of our present notation we shall skip the degenerate case as not sufficiently interesting and abbreviate $\varrho_n = \{\{L_n|D_n\}$. Assuming also that these overlaps do not vanish we may write the unit operator in $\mathcal{H}_{(\mathrm{R})}$ in the form of the decomposition

$$I = \sum_{n=0}^{\infty} |D_n\}\frac{1}{\varrho_n}\{\{L_n|. \tag{A.4}$$

The replacement of this operator by the projector Π of equation (51) proves trivial,

$$\Pi = \sum_{n=1}^{\infty} |D_n\}\frac{1}{\varrho_n}\{\{L_n|. \tag{A.5}$$

Finally, the spectral decomposition of the Hamiltonian remains also very natural and transparent,

$$\hat{H}^{(\mathrm{KG})} = \sum_{n=0}^{\infty} |D_n\}\frac{\varepsilon_n}{\varrho_n}\{\{L_n|. \tag{A.6}$$

The second item in equation (A.2) may also be written in its Hermitian conjugate form, i.e., as the relation

$$\hat{H}^{(\mathrm{KG})}G^{-1}|L_n\}\} = \varepsilon G^{-1}|L_n\}\}, \qquad \varepsilon = E^2 > 0. \tag{A.7}$$

Its comparison with the first item in the same equation reveals that only a multiplicative constant may distinguish between these two alternative forms of the eigenkets of the same operator,

$$|L_n\}\} = q_n G|D_n\}. \tag{A.8}$$

For the sake of simplicity, let us require that q_n (which might be called generalized quasi-parity) is real. Then our latter formula defines the left-action eigenvectors in terms of the right-action ones (or vice versa) in the halved Hilbert space. Now, it is important to imagine that our choice of the value of the quasi-parity q_n determines fully the above-introduced overlaps

$$\varrho_n = \{\{L_n|D_n\} \equiv q_n\{D_n|G|D_n\} \tag{A.9}$$

(and, in particular, their signs) because the values of the matrix elements $\{D_n|G|D_n\}$

themselves are real (the metric G is always Hermitian) and may be considered known (or calculated, by integration) in advance.

A.2. Larger Hilbert spaces $\mathcal{R}$

The two-component functions defined by equations (A.1) in the Hilbert spaces $\mathcal{R}$ may be used again as pre-multiplication factors which convert the subtracted matrix elements into another fundamental biorthogonality condition

$$0 = (\tau\sqrt{\varepsilon_m} - \sigma\sqrt{\varepsilon_n})\langle\langle m^{(\tau)}|n^{(\sigma)}\rangle. \tag{A.10}$$

As long as $\mathcal{R} = \mathcal{H} \oplus \mathcal{H}$, we could just insert and use the results of the preceding subsection. Alternatively, our notation enables us to shorten such a procedure and deduce that the overlaps $\langle\langle m^{(\tau)}|n^{(\sigma)}\rangle$ must necessarily vanish whenever the new energies $E_n^{(\pm)}$ remain non-degenerate. The latter condition means that the overlaps are zero not only when $\varepsilon_m \neq \varepsilon_n$, but also when the signatures do not coincide, i.e., whenever $\pm 1 = \tau \neq \sigma = \mp 1$.

In a way paralleling our previous text we abbreviate now $\mu_n^{(\tau)} = \langle\langle n^{(\tau)}|n^{(\tau)}\rangle$ and expand the unit operator in the larger space $\mathcal{R}$,

$$I = \sum_{n=0}^{\infty}\sum_{\tau=\pm 1}|n^{(\tau)}\rangle\frac{1}{\mu_n^{(\tau)}}\langle\langle n^{(\tau)}|. \tag{A.11}$$

With $E_n^{(\tau)} = \tau\sqrt{\varepsilon_n}$, the parallel spectral decomposition of the Hamiltonian reads

$$\hat{h}^{(\mathrm{PB})} = \sum_{n=0}^{\infty}\sum_{\tau=\pm 1}|n^{(\tau)}\rangle\frac{E_n^{(\tau)}}{\mu_n^{(\tau)}}\langle\langle n^{(\tau)}|. \tag{A.12}$$

The use of the pseudo-Hermiticity rules (4) + (5) transforms again the equation for the Hermitian conjugate double-bra vectors into the following equivalent equation for the double-kets:

$$\hat{h}^{(\mathrm{PB})}\mathcal{G}^{-1}|n^{(\tau)}\rangle\rangle = E\mathcal{G}^{-1}|n^{(\tau)}\rangle\rangle, \qquad E = \tau\sqrt{\varepsilon}. \tag{A.13}$$

This means that another multiplicative real constant $Q_n^{(\tau)}$ is to be introduced in order to distinguish between the eigenfunctions,

$$|n^{(\tau)}\rangle\rangle = Q_n^{(\tau)}\mathcal{G}|n^{(\tau)}\rangle. \tag{A.14}$$

This formula again implies that

$$\mu_n^{(\tau)} = \langle\langle n^{(\tau)}|n^{(\tau)}\rangle \equiv Q_n^{(\tau)}\langle n^{(\tau)}|\mathcal{G}|n^{(\tau)}\rangle. \tag{A.15}$$

After we recall all the definitions, we may already evaluate

$$\mu_n^{(\tau)} = 2\tau\sqrt{\varepsilon_n}\varrho_n. \tag{A.16}$$

In conclusion let us return to our initial assumptions and try to admit that the ground-state-like eigenvalue $\varepsilon_0 = 0$ vanishes. Then, equation (9) would merely define two identical vectors at both signs $^{(\pm)}$ at $n = 0$. Such a feature is characteristic for the non-diagonalizable (usually called Jordan-block) limits of non-Hermitian operators [17]. In the spectra of differential operators these points are also known as 'Bender–Wu singularities' [24], as the points of an 'unavoided level-crossing' [8] or simply as 'exceptional points' [18]. In our present paper, their properties will only be derived via a limiting transition $\varepsilon_0 \to 0^+$ from the regular domain.

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